10.3. Bellman-Ford Algorithm

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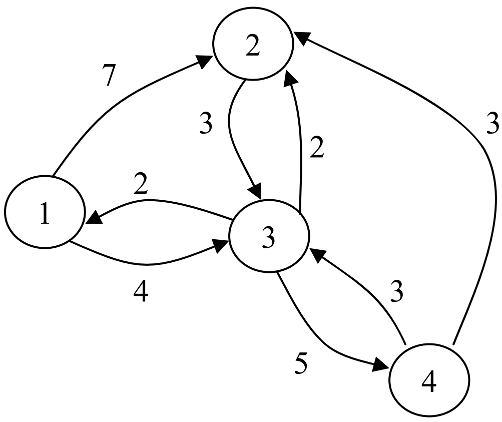
Basic problem 2

An oriented graph is given containing no contours with negative length (equal to the sum of weights of their arcs). Find the minimal paths from a given node to all other nodes of the graph.

This problem is a natural summary of the network planning problem. In this case minimal paths need to be found not only within a network, but also for a random path starting from a given node of the graph. One of the best-known algorithms for solving this dynamic optimization problem is Bellman-Ford's. It uses output information from distances matrix R, introduced earlier in paragraph 10.1.

First we will consider a simple example which demonstrates the idea of Bellman-Ford's algorithm.

Example 3. In fig. 5 is shown an oriented weighted graph with 4 nodes, containing 3 contours. Find the shortest distances from node 1 to all the other nodes.



$$R = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} 0 & 7 & 4 & \infty \\ \infty & 0 & 3 & \infty \\ 2 & 2 & 0 & 5 \\ \infty & 3 & 3 & 0 \end{bmatrix}$$

Fig. 5. Graph with 3 contours and its corresponding distances matrix.

Solution. The variable D_i stands for the sought minimal distance from node 1 to the i-th node. The solution goes through stages with and every stage includes a new node and the new value for D_i is recalculated. This goes on until the resulting values can no longer get smaller. Variable k stands for the number of the stage. In the initial stage $k = 1$ all D_i are equal to the direct distance from node 1 to the corresponding node.

We have:

$k = 1$, $\quad D_i = R_{1i}$, i.e. $D_1 = 0$, $D_2 = 7$, $D_3 = 4$, $D_4 = \infty$.

$k = 2$, $\quad D_1 = 0$,

D_2 = min from D_2 and the sums of the remaining D_i with the distances from the i-th node to the second node (second column of the matrix). We have:

$D_2 = \min\{7, 0+7, \underline{4+2}, \infty+3\} = 6$, (the minimum is underlined).

Likewise for a minimum of D_3 and the sums of the remaining D_i with the third column of R:

$D_3 = \min\{4, \underline{0+4}, 7+3, \infty+3\} = 4$.

Likewise for D_4 we calculate: Аналогично

$D_4 = \min\{\infty, 0+\infty, 7+\infty, \underline{4+5}\} = 9$.

$k = 3$, $D_1 = 0$,

$D_2 = \min\{6, 0+7, \underline{4+2}, 9+3\} = 6$.

$D_3 = \min\{4, \underline{0+4}, 6+3, 9+3\} = 4$.

$D_4 = \min\{9, 0+\infty, 6+\infty, \underline{4+5}\} = 9$.

We have reached a stage at which the results are the same as in the previous one and they can't get better. Therefore we've reached the minimal distances that we sought. The calculations are written in the following table:

k	D_1	D_2	D_3	D_4
1	0	7	4	∞
2	0	6	4	9
3	0	6	4	9

The last row shows the minimal paths from 1 to every node.

General case - formulas for Bellman-Ford's algorithm for graph with *n* nodes

For the sake of convenience we consider the node for which minimal distances are sought as number 1.

$k = 1$, $D_i = R_{1i}$, $i = 1, 2, \ldots, n$.

$k = 2$, $D_1 = 0$,

(1) $D_2 = \min\{D_2, D_1+R_{12}, D_3+R_{32}, \ldots, D_n+R_{n2}\}$,

$D_3 = \min\{D_3, D_1+R_{13}, D_2+R_{23}, \ldots, D_n+R_{n3}\}$,

...

$D_n = \min\{D_n, D_1+R_{1n}, D_2+R_{2n}, \ldots, D_{n-1}+R_{n-1,n}\}$.

$k = 3, 4, \ldots, n-2$.

This procedure is repeated until we get two identical rows of distance in the table or until n-2 is reached in the worst case.

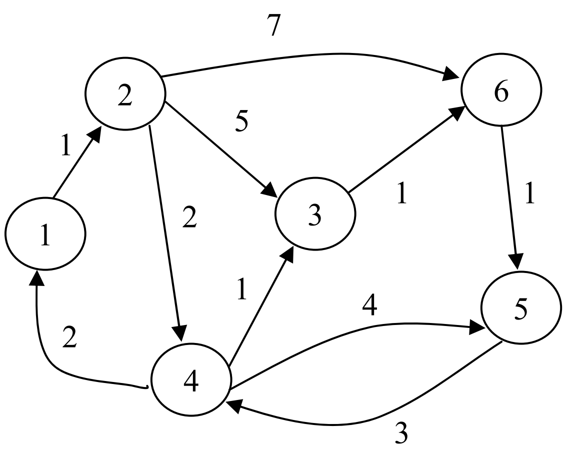
Description of Bellman-Ford's algorithm using metalanguage

Let a given oriented weighted graph (V, E) with n nodes V and arcs E, which does not contain a contour with negative length. The aim is to find the minimal distances from node number p to all other nodes of the graph. The distances matrix R is given and the sought distances are recorded in an array D. The number of stages obviously does not exceed n-2. Incoming and outgoing operations have been skipped.

```
begin
for v∈V do D[v]:=R[p,v]; D[p]:=0;
for k:=1 to n-2 do
    begin for v∈V \ {p} do
            for u∈V do D[v]:= min ( D[v], D[u] + R[p,v] )
    end
end
```

The number of necessary arithmetical operations is $O(n^3)$.

Example 4. Use Bellman-Ford's method for finding the minimal distances from node 1 to all other nodes of the graph shown in fig. 6.



$$R = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{bmatrix} 0 & 1 & \infty & \infty & \infty & \infty \\ \infty & 0 & 5 & 2 & \infty & 7 \\ \infty & \infty & 0 & \infty & \infty & 1 \\ 2 & \infty & 1 & 0 & 4 & \infty \\ \infty & \infty & \infty & 3 & 0 & \infty \\ \infty & \infty & \infty & \infty & 1 & 0 \end{bmatrix}$$

Fig. 6. Oriented graph with six contours.

Solution: Using formulas (1) we write the results down in a table. In the formulas the case of a medial minimum has been underlined.

k =1, $D_1 = 0$, $D_2 = R_{12} = 1$, $D_3 = D_4 = D_5 = D_6 = \infty$.

k =2, $D_1 = 0$,

$D_2 = \min \{D_2, D_1+R_{12}, D_3+R_{32}, D_4+R_{42}, D_5+R_{52}, D_6+R_{62}\} =$
$\min \{1, \underline{0+1}, \infty+\infty, \infty+\infty, \infty+\infty, \infty+\infty\} = 1$,

$D_3 = \min \{D_3, D_1+R_{13}, D_2+R_{23}, D_4+R_{43}, D_5+R_{53}, D_6+R_{63}\} =$
$\min \{\infty, 0+\infty, \underline{1+5}, \infty+1, \infty+\infty, \infty+\infty\} = 6$,

$D_4 = \min \{D_4, D_1+R_{14}, D_2+R_{24}, D_3+R_{34}, D_5+R_{54}, D_6+R_{64}\} =$

$\min\{\infty, 0+\infty, \underline{1+2}, 6+\infty, \infty+3, \infty+\infty\} = 3,$

$D_5 = \min\{D_5, D_1+R_{15}, D_2+R_{25}, D_3+R_{35}, D_4+R_{45}, D_6+R_{65}\} =$

$\min\{\infty, 0+\infty, 1+\infty, 6+\infty, \underline{3+4}, \infty+1\} = 7,$

$D_6 = \min\{D_6, D_1+R_{16}, D_2+R_{26}, D_3+R_{36}, D_4+R_{46}, D_5+R_{56}\} =$

$\min\{\infty, 0+\infty, 1+7, \underline{6+1}, 3+\infty, 7+\infty\} = 7.$

$k=3$, $D_1 = 0,$

$D_2 = \min\{1, \underline{0+1}, 6+\infty, 3+\infty, 7+\infty, 7+\infty\} = 1,$

$D_3 = \min\{6, 0+\infty, 1+5, \underline{3+1}, 7+\infty, 7+\infty\} = 4,$

$D_4 = \min\{3, 0+\infty, \underline{1+2}, 4+\infty, 7+3, 7+\infty\} = 3,$

$D_5 = \min\{7, 0+\infty, 1+\infty, 4+\infty, \underline{3+4}, 7+1\} = 7,$

$D_6 = \min\{7, 0+\infty, 1+7, \underline{4+1}, 3+\infty, 7+\infty\} = 5.$

$k=4$, $D_1 = 0,$

$D_2 = \min\{1, \underline{0+1}, 6+\infty, 3+\infty, 7+\infty, 7+\infty\} = 1,$

$D_3 = \min\{6, 0+\infty, 1+5, \underline{3+1}, 7+\infty, 7+\infty\} = 4,$

$D_4 = \min\{3, 0+\infty, \underline{1+2}, 4+\infty, 7+3, 7+\infty\} = 3,$

$D_5 = \min\{7, 0+\infty, 1+\infty, 4+\infty, \underline{3+4}, 7+1\} = 7,$

$D_6 = \min\{7, 0+\infty, 1+7, \underline{4+1}, 3+\infty, 7+\infty\} = 5.$

$k=5$, $D_1 = 0,$

$D_2 = \min\{1, \underline{0+1}, 6+\infty, 3+\infty, 7+\infty, 5+\infty\} = 1,$

$D_3 = \min\{6, 0+\infty, 1+5, \underline{3+1}, 7+\infty, 5+\infty\} = 4,$

$D_4 = \min\{3, 0+\infty, \underline{1+2}, 4+\infty, 7+3, 5+\infty\} = 3,$

$D_5 = \min\{7, 0+\infty, 1+\infty, 4+\infty, 3+4, \underline{5+1}\} = 6,$

$D_6 = \min\{7, 0+\infty, 1+7, \underline{4+1}, 3+\infty, 5+\infty\} = 5.$

k	D_1	D_2	D_3	D_4	D_5	D_6
1	0	1	∞	∞	∞	∞
2	0	1	6	3	7	7
3	0	1	4	3	7	5
4	0	1	4	3	6	5
5	0	1	4	3	6	5